PREFACE

Understanding & Harnessing Wavelet “Elephants”

It was six men of Hindustan
To learning much inclined,
Who went to see the Elephant
(Though all of them were blind)
That each by observation
Might satisfy the mind.

The first approached the Elephant
And happening to fall
Against his broad and sturdy side
At once began to bawl:
"Bless me, it seems the Elephant
Is very like a wall".

The second, feeling of his tusk,
Cried, "Ho! What have we here
So very round and smooth and sharp?
To me 'tis mighty clear
This wonder of an Elephant
Is very like a spear".

The third approached the animal,
And happening to take
The squirming trunk within his hands,
Then boldly up and spake:
"I see," quoth he, "the Elephant
Is very like a snake."

The Fourth reached out an eager hand,
And felt about the knee.
"What most this wondrous beast is like
Is mighty plain," quoth he;
"'Tis clear enough the Elephant
Is very like a tree!"

The Fifth, who chanced to touch the ear,
Said: "E'en the blindest man
Can tell what this resembles most;
Deny the fact who can,
This marvel of an Elephant
Is very like a fan!"

The Sixth no sooner had begun
About the beast to grope,
Than, seizing on the swinging tail
That fell within his scope,
"I see," quoth he, "the Elephant
Is very like a rope!"

And so these men of Hindustan
Disputed loud and long,
Each in his own opinion
Exceeding stiff and strong,
Though each was partly in the right
And all were in the wrong.

"*The Six Blind Men and the Elephant*" — A Poem by John Godfrey Saxe (1816 – 1887)

The subject of wavelets is certainly an "Elephant"—besides being *extremely powerful*, it can also become *extremely "heavy"* in terms of math. And the various methods of looking at the subject are staunchly defended by the various authors "*Exceeding stiff and strong*".

There are a lot of ways to look at the wavelet "elephant". Vector Spaces, Function Spaces, Frame Theory, Set Theory, Matrices and Transposes, Finite Elements, Con tinuous Time Representations (with double infinite integrals), extensions of the Fast Fourier Transform or the Short-Time Fourier Transform and even a curve on a sphere in 4-dimensional space—all these are used by various authors with the same evangelical zeal as the learned men of Indostan. None of these are really "wrong",

How This Book Differs From Other Wavelet Texts

"*Conceptual Wavelets in Digital Signal Processing*", however, is vastly different from other books in that we use numerous examples, figures, and demonstrations to show how to understand and use wavelets. This is a very complete and in-depth treatment of the subject, but from an intuitive, conceptual point of view. We let you look at a few key equations found in the more mathematically oriented texts—but only *after* the concepts are demonstrated and understood.* Then if you desire further study from traditional texts, this allows you to recognize these equations and understand in advance how they relate to the real world having actually seen them "in action".

It has been gratifying to present the 3-day course "Wavelets: A Conceptual, Practical Approach" at universities, corporations, and conference centers around the country for the past few years. Much of this book is "built" on these slides and improved by the comments and suggestions from the attendees. Those with little or no math background have expressed gratitude for being able to "see the elephant" enough to understand it and use it's power. Those with a strong mathematical background have expressed thanks for new insights and intuitive understanding that was not immediately evident from the equations.

One of the principle contributions of wavelets has been to bring those academic fields together to observe the "elephant" and to "satisfy the mind". It is not surprising then that you will find this particular pachyderm described in terms of *wavelets, wavelet filters, wavelet transforms, filter banks, multirate systems, matched filtering, multiresolution analysis* and so on.

This author's background is in Digital Signal Processing (Fast Fourier Transforms, Digital Filtering, etc.). and the description of the elephant is no doubt biased toward time or frequency representations of data. You will soon learn, however, that both the *power* and the *complexity* of wavelets lies in the fact that they deal with (are localized in) <u>*both* time *and* frequency</u>! It is especially important to understand that this dual (time/frequency) nature adds literally another dimension to wavelets! Instead of the data being shown as a function of time <u>*or*</u> as a function of frequency we now can look

* The occasional equation, if especially relevant to the explanations on a particular page, can also be found in the footnotes on that page.

at the data simultaneously in terms of time *and* frequency (or at least effective frequency).

Is the extra dimension of effort in learning to use these wavelet tools worth it? I believe the answer from all the authors would be a resounding "Yes!!". They, like this author, have seen how powerful and how handy this "animal" can be for processing signals or images that have "events" (changes in amplitude, frequency or shape) that start and stop. These "non-stationary" signals—the most interesting kind—are as diverse as a human heartbeat, a telemetry pulse from a missile (friendly or not), earthquakes seismic data, and the financial patterns in the stock market.

Because of all this tremendous capability, most authors (including myself) sincerely try to prevent the student from becoming discouraged when facing this extra dimension. Thus you will see terms in the title of many wavelet books and websites such as "Gentle", "Tutorial", "Friendly", "*Really* Friendly", "A Primer", "Made Easy", and even "For Kids" (we would like to meet these child prodigies!).

This is also why authors will teach using the tools with which they are most familiar—and for most wavelet writers this is *applied mathematics* (a look inside traditional wavelet texts quickly reveals a heavy dependence on math). This book takes a very different approach in that it *doesn't* rely on proofs, theorems, lemmas, etc. etc. to try to teach *concepts* through *equations*. We emphasize informed *use* of wavelets and leave the rigorous *proofs* to scholarly texts. In the appendix, we reference some excellent traditional mathematics-based texts, articles, and websites for additional study if desired.

This author also feels strongly (no lack of evangelical zeal here!) that it is important to be sure that wavelet data is not misused or misinterpreted. As with any technology, blindly following equations out of context without understanding the concepts behind them can lead to misinformation. We will show you how to "read" the results of wavelet displays correctly. We point out common pitfalls in wavelet transforms and how to avoid them. The real-world intuitive understanding you will obtain from reading this book should allow you to take full advantage of the powerful capability of wavelets with confidence in obtaining true and meaningful results and without fear of degradation of the data.

How This Book Is Laid Out— Study Suggestions

(How do you eat an elephant?—one bite at a time).

So how do you "digest" this book? One chapter at a time. Each chapter is designed to build on the previous chapters to help you gain a conceptual understanding of wavelets.

Chapter One presents an overview of wavelets, wavelet filters and wavelet transforms and shows a little of what they can do. This should put you way ahead of the "six blind men from Indostan" by providing a good "peek at the pachyderm". The familiar FFT/DFT is first reviewed and then compared to the Continuos Wavelet Transform (CWT). The conventional (decimated) Discrete Wavelet Transform (DWT) and the Undecimated DWT (UDWT) are introduced and compared. Examples of the capabilities of these transforms are shown, along with a short overview of the various types of wavelets

For a first "bite", you might also want to look at a short article "Wavelets: Beyond Comparison" written by the author as a staff tutorial for Applied Technology Institute. (**www.aticourses.com/ati_tutorials.htm)**.

Chapter Two provides a step-by-step walk-through of the Continuous Wavelet Transform using a very simple example of 8 exam scores and a Haar wavelet. We actually construct a CWT display from this data to learn how to "read" this type of display.

Chapter Three uses the same example of 8 exam scores and the Haar wavelet filters, but provides a step-by-step walk through of the Undecimated Discrete Wavelet Transform (UDWT—a.k.a. RDWT, A' Trous, or Shift Invariant). We show how this is very similar to the Continuous Wavelet Transform from Chapter Two.

Chapter Four highlights the downasmpling and upsampling that is added to the UDWT to produce the better-known conventional (decimated) DWT. We continue with the same example of 8 exam scores and the Haar wavelet in another step-by-step walk-through. We also show some simple compression and de-noising and take our first look at some DWT displays.

Chapter Five shows how some wavelet filters ("crude" wavelets) are generated using an explicit mathematical equation. The equations, though very simple, are continuous in time and so we show how to get the discrete points needed to produce actual digital wavelet filters of varying length.

If Chapter Five can be thought of as "Filters from Wavelets", then Chapter Six can be thought of as "Wavelets from Filters". Starting with wavelet filters that have very few points (only 2 for the Haar), we learn how we can interpolate or "stretch" them to hundreds of points. Then we learn how to use these very long filters to produce filters of *any* desired length. In other words we go from "Filters to Wavelets to Filters".

In Chapter Seven we explore further and compare the 3 major types of wavelet transforms—the Continuous Wavelet Transform (CWT), the Undecimated Discrete Wavelet Transform (UDWT), and the conventional (decimated) Discrete Wavelet Transform

(DWT). We also look briefly at the Wavelet Packet Transform (WPT). We examine the strengths and weaknesses of each type and show the general types of application of each. We compare and relate each type of transform to the others and show how, in each type, that we are still comparing data with the various wavelet filters.

Now somewhat familiar with the wavelet transforms, in Chapter Eight we look at what gives the “elephant” such strength and power—the Perfect Reconstruction Quadrature Mirror Filters (PRQMF). We not only discover the amazing “mirror” relationships these filters have to each other, but that they are actually factors of the relatively simple Halfband Filters—the very Heart and Soul of the wavelet “elephant” We demonstrate *orthogonality* in wavelet filters by comparing them to simple Cartesian coordinates (x-y-z). In the last section we show how the halfband filters can also be factored another way into *biorthogonal* wavelets filters. There is some mathematics (mostly at the high-school algebra level) in this chapter, but this is in line with a major goal of the book—to introduce some key equations found in conventional wavelet literature after providing an intuitive understanding of the concepts (in this case *spectral factorization*).

In Chapter Nine we introduce a few more desirable qualities of the various wavelets—some by comparing with arbitrary or “fake” wavelets to highlight these qualities. We will demonstrate such concepts as *regularity, vanishing moments,* and stretching and sliding the wavelet filters to “*match*” the hidden event in a signal (and thus determine its location, frequency, and general shape). We also talk about the adaptability of wavelets and about spending too much effort to find the “perfect” set of wavelet filters (the sport of basis hunting). We then demonstrate an even easier way to find the “magic numbers” of the filters by using the above desirable qualities to create some simple equations and then using direct substitution to solve them.

By Chapter Ten, we have learned much about the properties of the various wavelets and wavelet filters and how they can (or cannot) be used in the various wavelet transforms. We now proceed to look at the major wavelet families and how the specific properties lead to some practical applications. We first look at some *crude wavelets* such as the Mexican Hat, Morlet, Gaussian, and Meyer. We then examine the complex versions of some of these crude wavelets (and some others) such as the Complex Shannon (Sinc), Complex Frequency B-Spline, Complex Morlet and the Complex Gaussian. We proceed to the *orthogonal* wavelets such as the Haar, Daubechies Family, Symlets, Coiflets, and the Discrete Meyer. Finally, we look at the *Biorthogonal* and *Reverse Biorthogonal* wavelets. In the last section we present a table summarizing the attributes of these various wavelets.

Chapter 11 gives you a break from examining the various aspects of the elephant and lets you “hop on the Howdah” (seat affixed to an elephant’s back) for a little “ride”. In this chapter we present some case studies of the applications of various wavelets to some real-life problems. We show how to separate noise from the signal by matching various wavelets to *either* the noise or the signal and then modifying the Discrete Wavelet Transform (DWT} results to keep only the “clean” signal. We also revisit the Continuous Wavelet Transform (CWT) to demonstrate its power. We perform compression and denoising on the classic “Barbara” image and even show how to remove “freckles” (skin imperfections). We highlight a pathological case where the Undecimated DWT will outperform the conventional (decimated) DWT and provide suggestions for insuring the integrity of the data. In all these “non-stationary” examples we show why an FFT (or an STFT) would not work as well—or not work at all.

In Chapter 12 we learn how a conventional DWT can downsample repeatedly and still have *alias cancellation* (if done correctly). We look in more detail at the alias cancellation capability of the Perfect Reconstruction Quadrature Mirror Filters from Chapter Eight. Perfect Reconstruction and Alias Cancellation are demonstrated in both the time and the frequency domains. In the last section we look at some equations found in much of the conventional literature that describe Alias Cancellation and Perfect Reconstruction (often called “No Distortion”). We relate these equations to the concepts we have already learned in this chapter.

The final chapter, Chapter Thirteen, clarifies some additional key equations from the traditional wavelet literature by explaining the concepts behind them and/or demonstrating some alternative methods. In particular, the (continuous time) Wavelet Function and Scaling Function Dilation Equations are examined. We show how the same results are obtained by the process of repeated upsampling and convolution. We then show how this upsampling/convolution process can produce artifacts in a conventional DWT that look like the “Wavelet Function” or “Scaling Function” itself! (don’t miss this one!). In the final section we explain some other terms found in much of the wavelet literature such as *forward DWT, inverse DWT, Fast Wavelet Transform, Fast Inverse Wavelet Transform*, and *Wavelet Domain* and how they relate to the intuitive concepts and terms we have learned in this book.

The Appendices are not “required reading” *per se*, but are included to provide enrichment and help. Appendix A presents additional ways to relate wavelet transforms to the more familiar Fourier Transforms and conventional DSP Filtering. Appendix B discusses Heisenberg Boxes and the Heisenberg Uncertainty Principle as applied to wavelets. Despite the Einstein-sounding name this is actually relativity – er – *relatively* simple (sorry, couldn’t resist). Appendix C is a reprint of a short 7-page article “Wavelets: Beyond Comparison” by the author. This simple “Staff Tutorial” written for the Applied Technology Institute (Riva, Maryland) is also included here for your convenience Appendix D is a resource that presents the author’s recommendations for wavelet books (at all levels), wavelet articles, and wavelet websites.

As you can see, the chapters build upon each other to allow you to observe and experience “this marvel of (a wavelet) elephant” It is sincerely hoped that with the concepts learned in these chapters and (if needed) some additional specific application-oriented literature and software, you will be able to harness this “mammoth” power to allow you a much more complete understanding of your data and to work with it in a more efficient and cost-effective manner.

“An investment in knowledge pays the best interest.”
—Benjamin Franklin